Journal of Mathematical Sciences: Advances and Applications
Volume 6, Number 1, 2010, Pages 177-192

PENALTY FUNCTIONS FOR THE MULTIOBJECTIVE OPTIMIZATION PROBLEM

DAU XUAN LUONG and TRAN VAN AN

Department of Natural Sciences
Quang Ninh Teacher Training College
Quang Ninh
Vietnam
e-mail: dauxuanluong@gmail.com

Department of Mathematics
Vinh University
182 Le Duan, Vinh City
Vietnam

Abstract

In this paper, we apply the penalty function method to the multiobjective optimization problem, in order to transform a constrained problem, referred to as the original problem, into a sequence of simpler constrained or unconstrained problems, referred to as the penalized problems. We show that any cluster point of a sequence of weak efficient solutions of the penalized problems is a weak efficient solution of the original problem. Moreover, under certain assumptions on the feasible region D and the objective function $\boldsymbol{f}$, we can show that every penalized problem has a weak efficient solution, and that a sequence of weak efficient solutions of the penalized problems always has at least one cluster point.

1. Introduction

The penalty function method is often employed to transform a constrained problem into a sequence of simpler constrained (or even

2010 Mathematics Subject Classification: 90C29, 90C25, 65K05, 65K10.

Keywords and phrases: multiobjective optimization, nonlinear optimization, penalty function, weak efficient solution.

Received September 15, 2010

unconstrained) problems, so that a sequence of solutions of the simpler constrained problems converges to a solution of the constrained problem.

There have been extensive studies on how to apply the penalty function method to the nonlinear optimization problem (see, for instance, [3, 4, 10, 14]). The penalty function method has been also employed to solve the multiobjective optimization problem (see [5, 6, 9, 13]). In [9], the weak efficient solutions of the multiobjective optimization problem $\mathrm{MOP}(D, \boldsymbol{f})$ were studied, together with the exponential penalty functions. It was shown in [9] that if $\boldsymbol{x}$ is a cluster point of a sequence of weak efficient solutions of the penalized problems, and $\boldsymbol{x}$ is feasible (i.e., $\boldsymbol{x} \in D$), then $\boldsymbol{x}$ is a weak efficient solution of the original problem. The feasibility of $\boldsymbol{x}$ was assumed in [9]. We are here using the exterior penalty function, and are able to show the following: (1) if $\boldsymbol{x}$ is a cluster point of a sequence of weak efficient solutions of the penalized problems, then $\boldsymbol{x}$ is feasible; (2) such a point $\boldsymbol{x}$ is a weak efficient solution of the original problem; (3) under certain assumptions imposed on the objective function $\boldsymbol{f}$, every penalized problem has a weak efficient solution, and every sequence of such weak efficient solutions has a cluster point, which in turn is a weak efficient solution of the original problem.

The paper is organized as follows. Necessary definitions, notations, and some basis results on the multiobjective optimization problem are given in Section 2. Section 3 presents our main results. The paper is concluded in Section 4.

2. Preliminaries

For $\boldsymbol{x} = (x_1, \dots, x_k)^T \in \mathbb{R}^k$ and $\boldsymbol{y} = (y_1, \dots, y_k)^T \in \mathbb{R}^k$, we adopt the following conventions:

$$\boldsymbol{x} < \boldsymbol{y} \Leftrightarrow x_i < y_i, \ \forall i = 1, \dots, k,$$

$$\boldsymbol{x} \not< \boldsymbol{y} \Leftrightarrow \exists i : x_i \geq y_i.$$

Let $\| \boldsymbol{y} \|$ denote the Euclidean norm of $\boldsymbol{y}$, namely,

$$\| \boldsymbol{y} \| = \sqrt{\sum_{i=1}^{k} y_i^2}.$$

The inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{k} x_i y_i.$$

Let $\mathbb{R}_+^k = \{\boldsymbol{x} = (x_1, \dots, x_k)^T \in \mathbb{R}^k : x_i \geq 0, i = 1, \dots, k\}$. Let D be a nonempty subset of $\mathbb{R}^k$. We henceforth assume that D is closed and convex. We consider the following *D-constrained multiobjective optimization problem*

$$\mathrm{MOP}(D, \boldsymbol{f}): \quad \min_{\boldsymbol{x} \in D} \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x})),$$

where $f_i : \mathbb{R}^k \to \mathbb{R}$, $i = 1, \dots, r$, are arbitrary functions on $\mathbb{R}^k$. D is referred to as the *feasible region* of $\mathrm{MOP}(D, \boldsymbol{f})$. If $\boldsymbol{x} \in D$, then $\boldsymbol{x}$ is called a *feasible point* of $\mathrm{MOP}(D, \boldsymbol{f})$. If $D = \mathbb{R}^k$, then $\mathrm{MOP}(D, \boldsymbol{f})$ is called an *unconstrained multiobjective optimization problem*.

Definition 2.1. The point $\boldsymbol{x} \in D$ is called a *weak efficient solution* of $\mathrm{MOP}(D, \boldsymbol{f})$, if there does not exist $\boldsymbol{y} \in D$ satisfying $\boldsymbol{f}(\boldsymbol{y}) < \boldsymbol{f}(\boldsymbol{x})$.

Thus, $\boldsymbol{x} \in D$ is a weak efficient solution of $\mathrm{MOP}(D, \boldsymbol{f})$, if and only if for all $\boldsymbol{y} \in D$, there exists some index i such that

$$\boldsymbol{f}_i(\boldsymbol{y}) \geq \boldsymbol{f}_i(\boldsymbol{x}).$$

Suppose that $\boldsymbol{F} : \mathbb{R}^k \to \mathbb{R}^{r \times k}$ is a mapping, whose range is a set of $r \times k$ real matrices. The *D-constrained vector variational inequality problem* is defined as follows:

$$\mathrm{VVIP}(D, \boldsymbol{F}): \quad \text{Find } \boldsymbol{x} \in D \text{ such that } \boldsymbol{F}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) \not< 0, \ \forall \boldsymbol{y} \in D.$$

If $D = \mathbb{R}^k$, then $\mathrm{VVIP}(D, \boldsymbol{F})$ is called an *unconstrained vector variational inequality problem*. Let $\boldsymbol{F}_i : \mathbb{R}^k \to \mathbb{R}^k$, $i = 1, \ldots, r$ be the component functions of $\boldsymbol{F}$, i.e., $\boldsymbol{F}_i(\boldsymbol{x})$ is the i-th row of the matrix $\boldsymbol{F}(\boldsymbol{x})$. Then

$$\boldsymbol{F}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) = (\langle \boldsymbol{F}_1(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle, \ldots, \langle \boldsymbol{F}_r(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle)^T.$$

Therefore, $\boldsymbol{x} \in D$ is a solution of $\mathrm{VVIP}(D, \boldsymbol{F})$, if and only if for all $\boldsymbol{y} \in D$,

$$(\langle \boldsymbol{F}_1(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle, \ldots, \langle \boldsymbol{F}_r(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle)^T \not< 0,$$

or in other words, there exists some index i such that

$$\langle \boldsymbol{F}_i(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle \geq 0.$$

The following theorem establishes the relationship between the weak efficient solutions of a multiobjective optimization problem and the solutions of the corresponding vector variation inequality problem, under the assumption on the convexity and differentiability of the objective function.

Theorem 2.2 ([2]). *Let $\boldsymbol{f}$ be convex and differentiable, i.e., each component f_i of $\boldsymbol{f}$ is convex and differentiable. Then $\boldsymbol{x}$ is a weak efficient solution of* $\mathrm{MOP}(D, \boldsymbol{f})$, *if and only if $\boldsymbol{x}$ is a solution of* $\mathrm{VVIP}(D, \boldsymbol{f}')$, *where $\boldsymbol{f}'$ is the total derivative of $\boldsymbol{f}$.*

The following results on the existence of solutions of a multiobjective optimization problem are useful for us later.

Theorem 2.3 ([8]). *Let $\boldsymbol{f}$ be convex and differentiable. Furthermore, suppose that D is unbounded and there exists $\boldsymbol{a} \in D$ such that*

$$\lim_{\| \boldsymbol{y} \| \to +\infty, \boldsymbol{y} \in D} \langle \nabla f_i(\boldsymbol{y}), \boldsymbol{y} - \boldsymbol{a} \rangle > 0, \quad i = 1, \ldots, r.$$

Then $\mathrm{MOP}(D, \boldsymbol{f})$ *has a weak efficient solution.*

Definition 2.4 ([1]). The function $\boldsymbol{F}: \mathbb{R}^k \to \mathbb{R}^{r\times k}$ is said to be *monotone* on $\mathbb{R}^k$, if each of its components is monotone, i.e., $\langle \boldsymbol{F}_i(\boldsymbol{y}) - \boldsymbol{F}_i(\boldsymbol{x}),\ \boldsymbol{y} - \boldsymbol{x}\rangle \geq 0$ for all $i = 1, \ldots, r$.

Theorem 2.5 ([1]). *Let* $\boldsymbol{F}: \mathbb{R}^k \to \mathbb{R}^{r\times k}$ *be a continuous and monotone function on* $\mathbb{R}^k$. *Assume that*

(1) *D is bounded, or*

(2) $\boldsymbol{F}$ *is weak coercive on D, namely, there exists a vector* $\boldsymbol{s} \in \mathbb{R}_+^r$ *and a vector* $\boldsymbol{a} \in D$ *such that*

$$\lim_{\|\boldsymbol{y}\|\to+\infty,\, \boldsymbol{y}\in D} \langle \boldsymbol{s}^T \boldsymbol{F}(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a}\rangle = +\infty.$$

Then $\mathrm{VVIP}(D,\ \boldsymbol{F})$ *has a solution.*

Combining Theorems 2.2, 2.3, and 2.5, we deduce the following sufficient condition for the solvability of a multiobjective optimization problem.

Corollary 2.6. *Let* $\boldsymbol{f}$ *be convex and differentiable. Assume that one of the following holds*:

(1) *D is bounded*,

(2) *D is unbounded and there exists a vector* $\boldsymbol{s} \in \mathbb{R}_+^r$ *and a vector* $\boldsymbol{a} \in D$ *such that*

$$\lim_{\|\boldsymbol{y}\|\to+\infty,\, \boldsymbol{y}\in D} \sum_{i=1}^{r} s_i \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a}\rangle = +\infty,$$

(3) *D is unbounded and there exists a vector* $\boldsymbol{a} \in D$ *such that*

$$\lim_{\|\boldsymbol{y}\|\to+\infty,\, \boldsymbol{y}\in D} \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a}\rangle > 0, \quad i = 1, \ldots, r.$$

Then $\mathrm{MOP}(D,\ \boldsymbol{f})$ *has a weak efficient solution.*

Proof. Due to the convexity of f_i, it follows that ∇f_i is continuous and monotone (see, for instance, Corollary 25.5.1, [11]). Let $\boldsymbol{F} : \mathbb{R}^k \to \mathbb{R}^{r\times k}$ be the mapping with components ∇f_i, $i = 1, \dots, r$. Then $\boldsymbol{F}$ is continuous and monotone.

If D is bounded, by Theorem 2.5, $\mathrm{VVIP}(D, \boldsymbol{F})$ has a solution $\boldsymbol{x}$. Using Theorem 2.2, we deduce that $\boldsymbol{x}$ is also a weak efficient solution of $\mathrm{MOP}(D, \boldsymbol{f})$.

Suppose that D is unbounded. If the second property holds, then $\boldsymbol{F}$ is weak coercive on D. Therefore, by Theorem 2.5, $\mathrm{VVIP}(D, \boldsymbol{F})$ has a solution $\boldsymbol{x}$. Hence, again by Theorem 2.2, $\boldsymbol{x}$ is a weak efficient solution of $\mathrm{MOP}(D, \boldsymbol{f})$. If the third property holds, then by Theorem 2.3, $\mathrm{MOP}(D, \boldsymbol{f})$ has a weak efficient solution. □

3. The Multiobjective Optimization Problem and the Penalty Functions

Firstly, we define and study the penalized problems.

3.1. The penalized problems

Definition 3.1. Let D be a nonempty subset of $\mathbb{R}^k$. A function $P : \mathbb{R}^k \to \mathbb{R}$ is called a *penalty function* for D, if it satisfies

$$\begin{cases} P(\boldsymbol{x}) = 0, & \boldsymbol{x} \in D, \\ P(\boldsymbol{x}) > 0, & \boldsymbol{x} \notin D. \end{cases} \tag{3.1}$$

In this paper, we assume that P is chosen so that it is not only convex, but also differentiable. For instance, if D is defined as

$$D = \{\boldsymbol{x} \in \mathbb{R}^k : g_j(\boldsymbol{x}) \leq 0,\ j = 1, \dots, m\}, \tag{3.2}$$

where $g_j : \mathbb{R}^k \to \mathbb{R}$, $j = 1, \dots, m$ are continuous functions, we can take

$$P(\boldsymbol{x}) = \sum_{j=1}^{m} [\max\{0, g_j(\boldsymbol{x})\}]^2. \tag{3.3}$$

It is straightforward to verify that P defined as above not only is convex and differentiable on $\mathbb{R}^k$, but also satisfies (3.1). Note that it is well-known that if P is convex, then

$$\langle \nabla P(\boldsymbol{x}),\ \boldsymbol{y} - \boldsymbol{x} \rangle \leq P(\boldsymbol{y}) - P(\boldsymbol{x}),$$

for every $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^k$.

Now, fix a set $K \supset D$. For $t > 0$, we define the following penalized problem

$$\mathrm{MOP}(K,\ \boldsymbol{f}^{(t)}): \quad \min_{\boldsymbol{x} \in K} \boldsymbol{f}^{(t)} = (\boldsymbol{f}_1^{(t)},\ \ldots,\ \boldsymbol{f}_r^{(t)}),$$

where $f_i^{(t)} = f_i + tP,\ i = 1, \ldots, r.$ The region K can well be $\mathbb{R}^k$, and in such a case, the penalized problem has no constraint. Next, we study the existence of solutions of the penalized problem $\mathrm{MOP}(K,\ \boldsymbol{f}^{(t)})$.

Lemma 3.2. Let $\boldsymbol{f} : \mathbb{R}^k \to \mathbb{R}^r$ *be convex and differentiable. Furthermore, assume that one of the following conditions holds*:

(1) K *is bounded*,

(2) K *is unbounded and there exists a vector* $\boldsymbol{s} \in \mathbb{R}_+^r$ *and a vector* $\boldsymbol{a} \in D$ *such that*

$$\lim_{\|\boldsymbol{y}\| \to +\infty,\ \boldsymbol{y} \in K} \sum_{i=1}^{r} s_i \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle = +\infty,$$

(3) K *is unbounded and there exists* $\boldsymbol{a} \in D$ *such that*

$$\lim_{\|\boldsymbol{y}\| \to +\infty,\ \boldsymbol{y} \in K} \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle > 0,\ i = 1,\ \ldots,\ r.$$

Then $\mathrm{MOP}(K,\ \boldsymbol{f}^{(t)})$ *has a weak efficient solution.*

Proof. It is clear that each $f_i^{(t)} = f_i + tP$ is convex and differentiable. If K is bounded, then by Corollary 2.6, $\mathrm{MOP}(K,\ \boldsymbol{f}^{(t)})$ has a weak efficient solution.

Suppose that the second condition is satisfied. We have

$$\begin{aligned}\sum_{i=1}^{r} s_i \langle \nabla f_i^{(t)}(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle &= \sum_{i=1}^{r} s_i \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle + t\sum_{i=1}^{r} s_i \langle \nabla P(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle \\ &\geq \sum_{i=1}^{r} s_i \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle + t(P(\boldsymbol{y}) - P(\boldsymbol{a})) \\ &\geq \sum_{i=1}^{r} s_i \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle \\ &\to +\infty, \text{ as } \|\boldsymbol{y}\| \to +\infty,\ \boldsymbol{y} \in K.\end{aligned}$$

The second inequality is due to the fact that $P(\boldsymbol{y}) \geq 0$ and $P(\boldsymbol{a}) = 0$ as $\boldsymbol{a} \in D$. Therefore, Corollary 2.6 implies that $\mathrm{MOP}(K, \boldsymbol{f}^{(t)})$ has a weak efficient solution.

Suppose that the third condition is satisfied. For each $i = 1, \dots, r$, we have

$$\begin{aligned}\langle \nabla f_i^{(t)}(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle &= \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle + t\langle \nabla P(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle \\ &\geq \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle + t(P(\boldsymbol{y}) - P(\boldsymbol{a})) \\ &\geq \langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle.\end{aligned}$$

Therefore,

$$\lim_{\|\boldsymbol{y}\| \to +\infty,\ \boldsymbol{y} \in K} \langle \nabla f_i^{(t)}(\boldsymbol{y}),\ \boldsymbol{y}-\boldsymbol{a}\rangle > 0.$$

Again by Corollary 2.6, we conclude that $\mathrm{MOP}(K, \boldsymbol{f}^{(t)})$ has a weak efficient solution. □

3.2. The convergence theorems

Let S and $S(t)$ denote the solution sets of $\mathrm{MOP}(D, \boldsymbol{f})$ and $\mathrm{MOP}(K, \boldsymbol{f}^{(t)})$, respectively. Let $\{t_n\}_n$ be a sequence of positive real numbers, which monotonically tends to $+\infty$ as $n \to +\infty$.

Lemma 3.3. *Assume that* $\boldsymbol{f}$ *is continuous, and that* $\boldsymbol{x}^{(n)} \in S(t_n)$ *for all* $n \in \mathbb{N}$. *Suppose that* $\boldsymbol{x}$ *is a cluster point of* $\{\boldsymbol{x}^{(n)}\}_n$. *Then* $\boldsymbol{x} \in D$.

Proof. We prove this lemma by contradiction. Suppose that $\boldsymbol{x}$ is the limit of a subsequence $\{\boldsymbol{x}^{(n_m)}\}_m$ of $\{\boldsymbol{x}^{(n)}\}_n$, and that $\boldsymbol{x} \notin D$. Then $P(\boldsymbol{x}) > 0$ and hence $P(\boldsymbol{x}) > \varepsilon$ for some $\varepsilon > 0$. Take $\boldsymbol{y} \in D$. Since $\boldsymbol{x}^{(n_m)} \in S(t_{n_m})$, there exists i_{n_m} such that

$$f_{i_{n_m}}^{(t_{n_m})}(\boldsymbol{y}) \geq f_{i_{n_m}}^{(t_{n_m})}(\boldsymbol{x}^{(n_m)}).$$

Since $i_{n_m} \in \{1, 2, \ldots, r\}$, there exists an infinite sequence $\{i_{n_{m_\ell}}\}_\ell$ of indices, all of which have the same value, say $i_{n_{m_\ell}} = 1$, for all $\ell \in \mathbb{N}$. To simplify the notation, we assume that $i_{n_m} = 1$ for all $m \in \mathbb{N}$. Therefore, for all $m \in \mathbb{N}$, we have

$$f_1^{(t_{n_m})}(\boldsymbol{y}) \geq f_1^{(t_{n_m})}(\boldsymbol{x}^{(n_m)}). \tag{3.4}$$

Since $P(\boldsymbol{x}^{(n_m)}) \to P(\boldsymbol{x}) > \varepsilon$, for all m sufficiently large, we have $P(\boldsymbol{x}^{(n_m)}) > \varepsilon$. Hence, for m sufficiently large,

$$\begin{aligned} f_1^{(t_{n_m})}(\boldsymbol{x}^{(n_m)}) - f_1^{(t_{n_m})}(\boldsymbol{y}) &= f_1(\boldsymbol{x}^{(n_m)}) - f_1(\boldsymbol{y}) + t_{n_m}(P(\boldsymbol{x}^{(n_m)}) - P(\boldsymbol{y})) \\ &\geq f_1(\boldsymbol{x}^{(n_m)}) - f_1(\boldsymbol{y}) + t_{n_m}\varepsilon \\ &\to f_1(\boldsymbol{x}) - f_1(\boldsymbol{y}) + \infty = +\infty, \text{ as } m \to +\infty. \end{aligned}$$

This contradicts (3.4). Note that here $P(\boldsymbol{y}) = 0$ as $\boldsymbol{y} \in D$. □

The following theorem shows that if a sequence of weak efficient solutions of the penalized problems converges to a point $\boldsymbol{x}$, then $\boldsymbol{x}$ is also a weak efficient solution of the original problem. Note that if there exists some $n \in \mathbb{N}$ such that $\boldsymbol{x}^{(n)} \in S(t_n) \cap D$, then it is easy to verify that $\boldsymbol{x}^{(n)}$ is also a solution of $\mathrm{MOP}(D, \boldsymbol{f})$.

Theorem 3.4. *Assume that* $\boldsymbol{f}$ *is continuous, and that* $\boldsymbol{x}^{(n)} \in S(t_n)$ *for all* $n \in \mathbb{N}$. *Then any cluster point of the sequence* $\{\boldsymbol{x}^{(n)}\}_n$ *is a weak efficient solution of* MOP$(D, \boldsymbol{f})$.

Proof. We assume that $\boldsymbol{x}$ is a cluster point of the sequence $\{\boldsymbol{x}^{(n)}\}_n$. Let $\{\boldsymbol{x}^{(n_m)}\}_m$ be a subsequence of $\{\boldsymbol{x}^{(n)}\}_n$, which converges to $\boldsymbol{x}$. By Lemma 3.3, we already have $\boldsymbol{x} \in D$.

Suppose for contradiction that $\boldsymbol{x} \notin S$. Then, there exists $\boldsymbol{y} \in D$ satisfying

$$f_i(\boldsymbol{y}) < f_i(\boldsymbol{x}), \; i = 1, \ldots, r.$$

Since $\boldsymbol{x}^{(n_m)} \in S(t_{n_m})$, there exists i_{n_m} such that

$$f_{i_{n_m}}^{(t_{n_m})}(\boldsymbol{y}) \geq f_{i_{n_m}}^{(t_{n_m})}(\boldsymbol{x}^{(n_m)}).$$

Since $i_{n_m} \in \{1, 2, \ldots, r\}$, there exists an infinite sequence $\{i_{n_{m_\ell}}\}_\ell$ of indices, all of which have the same value, say $i_{n_{m_\ell}} = 1$, for all $\ell \in \mathbb{N}$. Again, to simplify the notation, we assume that the sequence $\{i_{n_m}\}_m$ itself satisfies this property, namely, $i_{n_m} = 1$, for all $m \in \mathbb{N}$. Therefore, for all $m \in \mathbb{N}$, we have

$$f_1^{(t_{n_m})}(\boldsymbol{y}) \geq f_1^{(t_{n_m})}(\boldsymbol{x}^{(n_m)}). \tag{3.5}$$

Since $f_1(\boldsymbol{y}) < f_1(\boldsymbol{x})$, we deduce that $f_1(\boldsymbol{y}) - f_1(\boldsymbol{x}) < -\varepsilon$, for some $\varepsilon > 0$. Since $\boldsymbol{x}^{(n_m)} \to \boldsymbol{x}$ as $m \to +\infty$, for sufficiently large m, we have

$$f_1(\boldsymbol{y}) - f_1(\boldsymbol{x}^{(n_m)}) < -\varepsilon.$$

Hence, for m sufficiently large, we have

$$f_1^{(t_{n_m})}(\boldsymbol{y}) - f_1^{(t_{n_m})}(\boldsymbol{x}^{(n_m)}) = f_1(\boldsymbol{y}) - f_1(\boldsymbol{x}^{(n_m)}) + t_{n_m}(P(\boldsymbol{y}) - P(\boldsymbol{x}^{(n_m)}))$$

$$< -\varepsilon - t_{n_m} P(\boldsymbol{x}^{(n_m)})$$

$$\leq -\varepsilon,$$

which contradicts (3.5). □

Theorem 3.5. *Let* $\boldsymbol{f} : \mathbb{R}^k \to \mathbb{R}^r$ *be convex and differentiable. Furthermore, assume that one of the following conditions holds*:

(1) *K is bounded*,

(2) *K is unbounded and there exists a vector* $\boldsymbol{a} \in D$ *such that*

$$\lim_{\| \boldsymbol{y} \| \to +\infty,\, \boldsymbol{y} \in K} \langle \nabla f_i(\boldsymbol{y}),\, \boldsymbol{y} - \boldsymbol{a} \rangle > 0,\ i = 1, \dots, r.$$

Assume also that $\boldsymbol{x}^{(n)} \in S(t_n)$ *for all* $n \in \mathbb{N}$. *Then, the sequence* $\{\boldsymbol{x}^{(n)}\}_n$ *has at least one cluster point, and every cluster point of this sequence is a weak efficient solution of* $\text{MOP}(D, \boldsymbol{f})$.

Proof. First note that by Lemma 3.2, we have $S(t_n) \neq \varnothing$. Therefore, the sequence $\{\boldsymbol{x}^{(n)}\}_n$ stated in the theorem is well-defined.

If K is bounded, then the sequence $\{\boldsymbol{x}^{(n)}\}_n \subseteq K$ is also bounded. Therefore, it has at least one cluster point. The claim that every cluster point of this sequence is a weak efficient solution of $\text{MOP}(D, \boldsymbol{f})$ follows directly from Theorem 3.4.

Now, assume that K is unbounded and there exists $\boldsymbol{a} \in D$ such that

$$\lim_{\| \boldsymbol{y} \| \to +\infty,\, \boldsymbol{y} \in K} \langle \nabla f_i(\boldsymbol{y}),\, \boldsymbol{y} - \boldsymbol{a} \rangle > 0,\ i = 1, \dots, r.$$

We aim to show that the sequence $\{\boldsymbol{x}^{(n)}\}_n$ is bounded. For $t > 0$, let $B(t)$ be the smallest closed ball in $\mathbb{R}^k$, centered at the origin, such that for all $\boldsymbol{y} \notin B(t)$, and for all $i = 1, \dots, r$, we have

$$\langle \nabla f_i^{(t)}(\boldsymbol{y}),\, \boldsymbol{y} - \boldsymbol{a} \rangle > 0,$$

or in other words,

$$\langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle + t\langle \nabla P(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle > 0.$$

Since

$$\langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle > 0,$$

when $\| \boldsymbol{y} \|$ is sufficiently large, and

$$t\langle \nabla P(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle \geq t(P(\boldsymbol{y}) - P(\boldsymbol{a})) \geq 0,$$

we deduce that $B(t)$ has finite radius.

Next, we show that $S(t) \subseteq B(t)$. Indeed, suppose on the contrary that there exists some $\boldsymbol{y} \in S(t) \setminus B(t)$. Then by definition of $B(t)$, for all $i = 1, \dots, r$, we have

$$\langle \nabla f_i^{(t)}(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle > 0,$$

or equivalently,

$$\langle \nabla f_i^{(t)}(\boldsymbol{y}),\ \boldsymbol{a} - \boldsymbol{y} \rangle < 0,\ i = 1, \dots, r.$$

Hence $\boldsymbol{y}$ is not a solution of $\mathrm{VVIP}(K, (\boldsymbol{f}^{(t)})')$. By Theorem 2.2, we deduce that $\boldsymbol{y} \notin S(t)$, a contradiction. Hence $S(t) \subseteq B(t)$.

On the other hand, for $t' > t$, we have $B(t') \subseteq B(t)$. Indeed, for all $\boldsymbol{y} \notin B(t)$ and for all $i = 1, \dots, r$, we have

$$\langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle + t\langle \nabla P(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle > 0,$$

which implies

$$\langle \nabla f_i(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle + t'\langle \nabla P(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle > 0, \tag{3.6}$$

as

$$\langle \nabla P(\boldsymbol{y}),\ \boldsymbol{y} - \boldsymbol{a} \rangle \geq P(\boldsymbol{y}) - P(\boldsymbol{a}) \geq 0.$$

By definition, $B(t')$ is the smallest ball such that for all $\boldsymbol{y} \notin B(t')$, the inequality (3.6) holds for all $i = 1, \ldots, r$. Therefore, $B(t')$ is contained in $B(t)$.

Finally, as $\{t_n\}_n$ is monotonically increasing, we have

$$B(t_1) \supseteq B(t_2) \supseteq \cdots \supseteq B(t_n) \supseteq \cdots.$$

Therefore, for all $n \in \mathbb{N}$, we have

$$\boldsymbol{x}^{(n)} \in S(t_n) \subseteq B(t_n) \subseteq B(t_1).$$

Since the radius of $B(t_1)$ is finite, we conclude that the sequence $\left\{\boldsymbol{x}^{(n)}\right\}_n$ is bounded. We complete the proof. □

Example 3.6. Let

$$D = \{\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 - x_1 - 1 \le 0, -x_2 + x_1^2 - 1 \le 0\}.$$

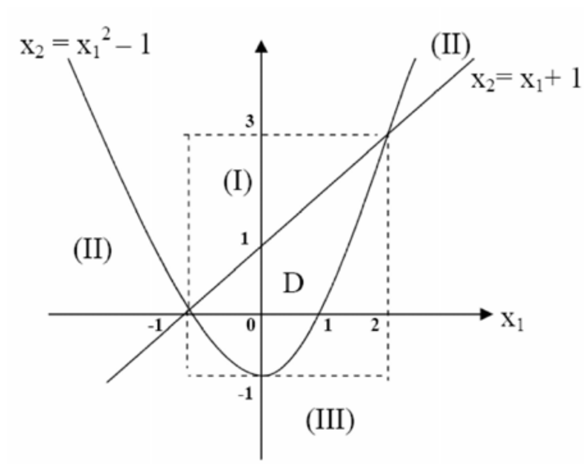

Figure 1. The feasible region D.

Take P as in (3.3):

$$P(\boldsymbol{x}) = [\max\{0,\ x_2 - x_1 - 1\}]^2 + [\max\{0,\ -x_2 + x_1^2 - 1\}]^2$$

$$= \begin{cases} 0, & \boldsymbol{x} \in D, \\ (x_2 - x_1 - 1)^2, & \boldsymbol{x} \in (\mathrm{I}), \\ (x_2 - x_1 - 1)^2 + (-x_2 + x_1^2 - 1)^2, & \boldsymbol{x} \in (\mathrm{II}), \\ (-x_2 + x_1^2 - 1)^2, & \boldsymbol{x} \in (\mathrm{III}). \end{cases}$$

We have

$$\nabla P(\boldsymbol{x}) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \boldsymbol{x} \in D, \\ \begin{pmatrix} -2(x_2 - x_1 - 1) \\ 2(x_2 - x_1 - 1) \end{pmatrix}, & \boldsymbol{x} \in (\mathrm{I}), \\ \begin{pmatrix} -2(x_2 - x_1 - 1) + 4x_1(-x_2 + x_1^2 - 1) \\ 2(x_2 - x_1 - 1) - 2(-x_2 + x_1^2 - 1) \end{pmatrix}, & \boldsymbol{x} \in (\mathrm{II}), \\ \begin{pmatrix} 4x_1(-x_2 + x_1^2 - 1) \\ -2(-x_2 + x_1^2 - 1) \end{pmatrix}, & \boldsymbol{x} \in (\mathrm{III}). \end{cases}$$

Let $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x})),$ where

$$f_1(\boldsymbol{x}) = \frac{x_1^2}{2} + \frac{x_2^2}{2} + x_1 x_2 + 2e^{x_2/2} - 2x_2,$$

$$f_2(\boldsymbol{x}) = e^{x_1} + x_1^2 - x_1 + \frac{x_2^2}{2} + 1.$$

Clearly, $\boldsymbol{f}$ chosen as above is convex and differentiable on $\mathbb{R}^2$. Let $K = \mathbb{R}^2$, and $\boldsymbol{a} = 0 \in D$. Then, we have

$$\langle \nabla f_1(\boldsymbol{y}),\ \boldsymbol{y} \rangle = (y_1 + y_2)^2 + y_2 e^{y_2/2} - 2y_2,$$

$$\langle \nabla f_2(\boldsymbol{y}),\ \boldsymbol{y} \rangle = 2y_1^2 + y_2^2 + y_1 e^{y_1} - y_1,$$

which are obviously positive when $\|\boldsymbol{y}\| \to +\infty$. Therefore, by Theorem 3.5, any sequence of weak efficient solutions of the penalized problems $\mathrm{MOP}(K,\ \boldsymbol{f}^{(t_n)})$, $n = 1, 2, \ldots,$ has a cluster point, and every cluster point of that sequence is a weak efficient solution of $\mathrm{MOP}(D,\ \boldsymbol{f})$.

4. Conclusion

We investigate the relationship between the set of weak efficient solutions of the original multiobjective optimization problem and the sets of weak efficient solutions of the corresponding penalized problems. Theorem 3.4 shows that every cluster point of a sequence of weak efficient solutions of the penalized problems is a weak efficient solution of the original problem, under only the continuity of $\boldsymbol{f}$. No convexity or differentiability is required. However, Theorem 3.5 requires such properties of $\boldsymbol{f}$. Several authors have established results on the existence of weak efficient solutions of a multiobjective optimization problem under less strict requirements imposed on $\boldsymbol{f}$, for instance, when $\boldsymbol{f}$ is nonsmooth and nonconvex (see, for instance, [7, 8, 12]). Based on these results, one may relax the assumptions imposed on the objective function $\boldsymbol{f}$ stated in Theorem 3.5.

References

[1] G. Y. Chen and X. Q. Yang, The vector complementary problem and its equivalences with the weak minimal element in ordered spaces, J. Math. Anal. Appl. 153 (1990), 136-158.

[2] G. Y. Chen and B. D. Craven, Existence and continuity of solutions for vector optimization, J. Math. Anal. Appl. 220 (1998), 90-98.

[3] R. Cominetti and J. P. Dussault, Stable exponential-penalty algorithm with super-linear convergence, J. Optim. Theory Appl. 83(2) (1994), 285-309.

[4] J. P. Evans and F. J. Gould, An existence theorem for penalty function theory, SIAM J. Control. 12 (1974), 505-516.

[5] X. Q. Huang and X. Q. Yang, Nonlinear Lagrangian for multiobjective optimization and applications to duality and exact penalization, SIAM J. Optim. 13(3) (2002), 675-692.

[6] X. Q. Huang, X. Q. Yang and K. L. Teo, Convergence analysis of a class of penalty methods for vector optimization problems with cone constraints, J. Global Optim. 36(4) (2006), 637-652.

[7] K. R. Kazmi, Existence of solutions for vector optimization, Appl. Math. Lett. 9 (1996), 19-22.

[8] G. M. Lee and D. S. Kim, Existence of solutions for vector optimization problems, J. Optim. Theory Appl. 81(3) (1994), 459-468.

[9] S. Liu and E. Feng, The exponential penalty function method for multiobjective programming problems, Optim. Methods Softw. 25(5) (2010), 667-675.

[10] V. H. Nguyen and J. J. Strodiot, On the convergence rate for a penalty function method of exponential type, J. Optim. Theory Appl. 27(4) (1979), 495-508.

[11] R. T. Rockafellar, Convex Analysis, Princeton Univ. Press, Princeton, New Jersey, 1970.

[12] L. B. Santos, G. Ruiz-Garzon, M. A. Rojas-Medar and A. Rufian-Lizana, Existence of weakly efficient solutions in nonsmooth vector optimization, Appl. Math. Comput. 200(2) (2008), 547-556.

[13] D. J. White, Multiobjective programming and penalty functions, J. Optim. Theory Appl. 13(3) (2002), 675-692.

[14] W. I. Zangwill, Nonlinear Programming: A Unified Approach, Prentice Hall, Englewood Cliffs, NJ, 1969.

■